APOS Theory

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REESE Project

- Track development of engineering student longitudinally
- Track transfer of ideas from math to physics to engineering
- Need a framework for describing conceptual development that covers several years worth of courses.

APOS Theory

- Says students build concepts through a standard set of steps
  - Action – Process – Object – Schema
- Built on ideas of Piaget (primarily by Dubinsky)
- Widely used in the RUME community

Definitions

- **Action**: able to carry out rote procedures, bound to specific representations
- **Process**: able to see the process as a whole, can use multiple representations, can reverse the process, compose with other processes, etc.
- **Object**: reify the process into an object, can discuss properties of the object or collections of examples of the process

Arithmetic

- **Action**: Can carry out rote computations
- **Process**: Can solve missing number problems
- **Object**: Can discuss properties (commutativity, associativity, etc.) of operations
- Algebra is at the Process/Object level

Function

- **Action**: A function is tied to a specific rule, or formula
- A student must perform each action
- The "answer" depends on the formula
- A student can only imagine a single value at a time (so \(x\) stands for a specific number)
- Algebra is at the Process/Object level
- **Process**: A function is a general input-output machine
- A student can imagine the entire process at once
- The process is independent of the formula
- The function is a transformation of the entire space at once

Oehrtman, Carlson, and Thompson, 2008
Function

**Action**
- Composition is substituting a formula for $x$.
- Inverse is about algebra or geometry (switch $x$ and $y$).
- Functions are conceived as static.
- A function’s graph is a geometric figure.

**Process**
- Composition is a coordination of two input-output processes.
- Inverse is the reversal of the process.
- Functions are conceived as dynamic where output changes with input.
- A graph defines a mapping of input to output values.

Oehrtman, Carlson, and Thompson, 2008

Function

**Object**
- Students can distinguish compositions (functions applied consecutively) from transforms (functions of functions).
- Students can work with function spaces (such as the solution space of a differential equation).
- Students understand multiple representations and properties of functions (e.g., even functions have cosine series expansions).

Accumulation (Integration)

**Action**
- Integration is a set of techniques (with a very odd notation).
- Integration finds a static area magically by computing an antiderivative.

**Process**
- Integration is adding many small contributions.
- Integrals measure an area that changes dynamically as the range of integration changes.

Inspired by Thompson and Silverman, 2008

Accumulation (Integration)

**Object**
- Integration is a function that takes a function as an input and produces another function as output.
- Integration transforms properties of functions in a standard way.

Expected Growth

- Pre-calculus students are typically at an Action level.
- Calculus students need to develop a Process level understanding to be successful.
- By Differential Equations students are starting to reach the Object level.
- Linear Systems students probably should be at the Object level.

Strengths of the Framework

- Widely used and understood (provides a good language for communication).
- Applicable to a broad range of concepts.
- Is useful in analyzing interview and other data.
Weaknesses of the Framework

- Transitions between levels are unclear
- Original descriptions very “Bourbaki”
- Schema is not well defined
- Conflates conceptual development (action to process) with what we think of as transfer (ability to use multiple representations)

Covariation

- Introduced by Carlson et. al. as a more natural (historical/physical) notion of function
- Think of a function as defining how two variables vary with each other rather than a set theoretic construct

Action to Process for Function

- Mental Action
  - Coordinate one variable with another
  - Coordinate direction of change
  - Coordinate amount of change
  - Coordinate average rate of change with uniform increments of input
  - Coordinate instantaneous rate of change

- Behavior
  - Labeling axes to show the two variables
  - Law of universal linearity (proportionality)
  - Plot secant lines and measure slope
  - Able to discuss how slopes change over uniform increments
  - Able to construct curve with clear concavity changes

Schema Development Details

- Intra – individual can make coherent connections between particular constructs (specific functions, etc.)
- Inter – individual is able to group items together and think of specific connections as examples of general ones
- Trans – a fully coherent schema including a sense of the limits of where the schema applies

- This is NOT nearly as widely used or accepted as general APOS theory

Transfer

- In APOS theory, the ability to coordinate two representations at once is often taken as a test of process-level understanding.
- While possessing such an ability may imply process level, our study asks if and when process level understanding leads to this ability.

Reaction Questions

Reactant: Dong-Hai Nguyen

- Why didn’t you mention Schema in the definitions of levels and the following examples?
- What level do you expect engineering students to possess and what level do they actually achieve in Calculus courses?
- Could you give an example on how we apply the APOS theory longitudinally along the REESE project?