APOS Theory

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REESE Project

- Track development of engineering student longitudinally
- Track transfer of ideas from math to physics to engineering

- Need a framework for describing conceptual development that covers several years worth of courses.

APOS Theory

- Says students build concepts through a standard set of steps
  Action – Process – Object – Schema

- Built on ideas of Piaget (primarily by Dubinsky)
- Widely used in the RUME community

Definitions

- **Action**: able to carry out rote procedures, bound to specific representations
- **Process**: able to see the process as a whole, can use multiple representations, can reverse the process, compose with other processes, etc.
- **Object**: reify the process into an object, can discuss properties of the object or collections of examples of the process

Arithmetic

- **Action**: Can carry out rote computations
- **Process**: Can solve missing number problems
- **Object**: Can discuss properties (commutativity, associativity, etc.) of operations
- Algebra is at the Process/Object level

Function

Action

- A function is tied to a specific rule, or formula
- A student must perform *each action*
- The "answer" depends on the formula
- A student can only imagine a single value at a time (so *x* stands for a specific number)

Process

- A function is a general input-output machine
- A student can imagine the *entire process* at once
- The process is independent of the formula
- The function is a transformation of the entire space at once

Oehrtman, Carlson, and Thompson, 2008

Function

Action
- Composition is substituting a formula for x
- Inverse is about algebra or geometry (switch x and y)
- Functions are conceived as static
- A function's graph is a geometric figure

Process
- Composition is a coordination of two input-output processes
- Inverse is the reversal of the process
- Functions are conceived as dynamic where output changes with input
- A graph defines a mapping of input to output values

Oehrtman, Carlson, and Thompson, 2008

Function

Object
- Students can distinguish compositions (functions applied consecutively) from transforms (functions of functions)
- Students can work with function spaces (such as the solution space of a differential equation)
- Students understand multiple representations and properties of functions (e.g. even functions have cosine series expansions)

Accumulation (Integration)

Action
- Integration is a set of techniques (with a very odd notation)
- Integration finds a static area magically by computing an antiderivative

Process
- Integration is adding many small contributions
- Integrals measure an area that changes dynamically as the range of integration changes

Inspired by Thompson and Silverman, 2008

Accumulation (Integration)

Object
- Integration is a function that takes a function as an input and produces another function as output
- Integration transforms properties of functions in a standard way

Expected Growth

- Pre-calculus students are typically at an Action level.
- Calculus students need to develop a Process level understanding to be successful.
- By Differential Equations students are starting to reach the Object level.
- Linear Systems students probably should be at the Object level.

Strengths of the Framework

- Widely used and understood (provides a good language for communication)
- Applicable to a broad range of concepts
- Is useful in analyzing interview and other data

Weaknesses of the Framework

- Transitions between levels are unclear
- Original descriptions very "Bourbaki"
- Schema is not well defined
- Conflates conceptual development (action to process) with what we think of as transfer (ability to use multiple representations)

Covariation

- Introduced by Carlson et. al. as a more natural (historical/physical) notion of function
- Think of a function as defining how two variables vary with each other rather than a set theoretic construct

Action to Process for Function

Mental Action	Behavior
Coordinate one variable with another	Labeling axes to show the two variables
Coordinate direction of change	Law of universal linearity (proportionality)
Coordinate amount of change	Plot secant lines and measure slope
Coordinate average rate of change with uniform increments of input	Able to discuss how slopes change over uniform increments
Coordinate instantaneous rate of change	Able to construct curve with clear concavity changes

Oehrtman, Carlson, and Thompson, 2008

Schema Development Details

- **Intra** – individual can make coherent connections between particular constructs (specific functions, etc.)
- **Inter** – individual is able to group items together and think of specific connections as examples of general ones
- **Trans** – a fully coherent schema including a sense of the limits of where the schema applies
- This is NOT nearly as widely used or accepted as general APOS theory

Transfer

- In APOS theory, the ability to coordinate two representations at once is often taken as a test of process-level understanding.
- While possessing such an ability may imply process level, our study asks if and when process level understanding leads to this ability.

Reaction Questions
Reactant: Dong-Hai Nguyen

- Why didn't you mention Schema in the definitions of levels and the following examples?
- What level do you expect engineering students to possess and what level do they actually achieve in Calculus courses?
- Could you give an example on how we apply the APOS theory longitudinally along the REESE project?