Resonant mode confinement in equilateral triangular dielectric cavities

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Abstract. The resonant optical modes of a high permittivity dielectric cavity with an equilateral triangular cross section are classified according to their possibility for confinement by total internal reflection (TIR). Eigenmode solutions of the scalar two-dimensional Helmholtz equation with Dirichlet boundary conditions, appropriate to a conducting boundary, are applied for this purpose. The individual plane wave components present in these modes are analyzed for their total internal reflection behavior and implied mode confinement when the conducting boundary is replaced by a dielectric mismatch. For the purely two-dimensional electromagnetic solutions, modes invariant under ±120° rotations cannot be confined for any dielectric mismatch; the confinement cutoffs for other modes are determined in terms of the modes’ quantum numbers. Improvement in TIR confinement by adjusting the longitudinal wavevector $k_z$ within a three-dimensional prism of triangular cross section is also described.


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1. Introduction

Micro-sized dielectric cavities are becoming increasingly important, due to their applications especially in microlasers and resonators with various geometries, including disks [1], triangles [2], squares [3, 4, 5], and hexagons [6, 7, 8]. Materials where the two-dimensional (2D) cross-section controls the properties include, for example, various cleaved semiconductor structures [2] as well as zeolite ALPO$_4$-5 crystals [6] and lasing semiconductor pyramids that grow by natural processes [9, 10]. For these materials, an understanding of the geometry dependence of the total internal reflection (TIR) that produces resonant modes can lead to improvements in designs of optical devices.

Here we consider mode confinement by TIR in equilateral triangular-based prisms; the triangular system is chosen here for its interesting symmetries and known analytic

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Helmholtz solution [11, 12, 13, 14, 15, 16], leading to simplifications in the analysis. For the reduced 2D electrodynamics (fields independent of a z-coordinate along an axis of symmetry), a simple analysis of the conditions for TIR is used as an alternative to the more powerful boundary element method [17]. As a first approximation, a 2D dielectric system with Dirichlet boundary conditions (DBC) is considered, taking the fields equal to zero at the boundary, equivalent to a metallic or conducting boundary. The exact analytic solutions for this 2D triangular system are reviewed; each mode is a superposition of six plane-wave components. The mode information is used to determine which modes will be TIR-confined if the dielectric is surrounded by a lower index dielectric (or vacuum), rather than a perfect conductor. Related analysis [18] shows that when a mode is strongly confined in the cavity by TIR, Dirichlet boundary conditions can be applied (approximately) to the relevant Helmholtz equation for both the TM and TE polarizations.

Obviously some error is made compared to employing a more technically complete solution of Maxwell’s equations such as boundary element methods [17]; the fields discussed here for the dielectric mismatch boundary are not close to the true fields unless the mismatch is very large. This approach, however, should approximately determine the symmetries of the confined modes and give reasonable estimates for the dielectric mismatch needed for their confinements. The improvement of confinement for finite–height prisms with the same cross sections, allowing for nonzero longitudinal wavevector $k_z$, is also discussed.

2. Simplified Quasi-Two-Dimensional Helmholtz Problems

In a cavity with electric permittivity $\epsilon$ and magnetic permeability $\mu$, we consider a scalar wave equation for any component $\psi$ of the electric or magnetic field,

$$\nabla^2 \psi - \frac{\epsilon \mu}{c^2} \frac{\partial^2}{\partial t^2} \psi = 0,$$

where $c$ is the speed of light in vacuum and $t$ is time. The cavity shape being considered is an equilateral triangular cross-section of edge $a$ in the $xy$-plane; it can be considered as relevant to a prism with the same cross section, under the assumption that the fields do not vary along the longitudinal direction ($z$). Ultimately, we want to discuss which modes will be confined in the cavity of index of refraction $n = \sqrt{\epsilon \mu}$ when it is surrounded by a uniform medium of lower index of refraction $n' = \sqrt{\epsilon' \mu'}$ (for example, vacuum).

We first analyze the modes assuming the fields go to zero at the cavity walls, using Dirichlet boundary conditions, which corresponds to perfectly conducting walls, or a mirrored cavity. (The range of applicability of these boundary conditions for a mode that is confined by TIR caused by an index mismatch, rather than reflecting boundaries, can be analyzed more thoroughly using Maxwell’s equations.) A solution at frequency $\omega$ is sought, with $e^{-i\omega t}$ time dependence, corresponding to 2D wavevector with magnitude $k = \frac{\omega}{c^*}$, where $c^* = \frac{c}{\sqrt{\epsilon \mu}}$ is the speed of light in the cavity medium. Then
we are solving the 2D eigenvalue problem (Helmholtz equation)

\[ \nabla^2_{xy} \psi = -k^2 \psi. \]  \hspace{1cm} (2)

In the general case, we are looking for a superposition of plane waves \( e^{i(\vec{k} \cdot \vec{r} - \omega t)} \), with the correct linear combination to satisfy the required boundary conditions.

We continue by discussing the modes in a 2D equilateral triangle, under the assumption of DBC for either the TM or TE polarizations.

2.1. Exact modes for an equilateral triangle with DBC

The triangular cross section offers an opportunity for exact solutions to the 2D Helmholtz equation. This problem has been solved analytically [11, 12, 13, 14] for both DBC and NBC in several contexts, including quantum billiards problems [16, 19, 20, 21], quantum dots [15], and lasing modes in resonators and mirrored dielectric cavities [2, 22]. In particular, it is interesting to note that only the triangles with angle sets \( \pi \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \), \( \pi \left( \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \right) \), and \( \pi \left( \frac{1}{2}, \frac{1}{3}, \frac{1}{6} \right) \) are classically integrable [20] and have simple closed form wavefunction solutions derived in various ways [12, 13, 21] following the first solution for triangular elastic membranes by Lamé [11]. Here we use the equilateral triangle of edge length \( a \) for its interesting symmetries and resulting simplifications.
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Figure 2. Scaled frequencies of some of the lowest possible modes in a 2D triangular system, as a function of quantum index \( m \), for indicated \( n > m \). Indices \( n \) and \( m \) must be integers of equal parity. \( c^* \) is the light speed in the medium.

Coordinates are used where the origin is placed at the geometrical center of the triangle, and the lower edge is parallel to the \( x \)-axis, see Fig. 1. The notation \( b_0, b_1, \) and \( b_2 \) is used to denote the lower, upper right, and upper left boundaries, respectively.

Following Chang et al. [2], some comments can be made on the sequence of reflections a plane wave trapped in the cavity makes, but with a slightly different physical interpretation. A plane wave leaving the lower face (\( b_0 \)) at angle \( \beta \) to the \( x \)-axis sequentially undergoes reflections at the other two faces, generating plane waves at angles \( 240^\circ - \beta, \beta - 120^\circ \), relative to the \( x \)-axis. Finally, when reflected again off the lower face, it comes out at \( 120^\circ - \beta \), which does not match the original wave unless \( \beta = 60^\circ \). However, when allowed to propagate again around the triangle, the sequence of angles is \( 120^\circ + \beta, -\beta \), which then comes out at \( +\beta \) after the reflection off the lower face. So the wave closes on itself after two full revolutions, no matter what the initial angle. As an example, the sequence of reflections starting with \( \beta = 70^\circ \) from boundary \( b_0 \) are shown in Fig. 1. Any original wave simply generates a set of six symmetry related waves rotated by \( \pm 120^\circ \) and inversions through the \( y \)-axis.

These considerations show that the general solution is a superposition of six plane waves, which can be obtained by 120-degree rotations of one partially standing wave. Consider initially a wavevector with \( xy \) Cartesian components, \( \vec{k} = (k_1, k_2) \), and defined basic vectors \( \vec{k}_1 = k_1 \hat{x}, \vec{k}_2 = k_2 \hat{y} \), producing a wavefunction written as

\[
\psi_0 = e^{i\vec{k}_1 \cdot \vec{r}} \sin \left[ \frac{\vec{k}_2 \cdot \vec{r} + k_2 a}{2\sqrt{3}} \right]
\]  

(3)

This is a combination of plane waves at two angles \( \beta \) and \(-\beta \) to the \( x \)-axis, where
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(a) mode 1
ground state
m=0, n=2
\( \omega_{a/c^*} = 7.2552 \)

(b) mode 4
2nd excited state
(nondegenerate)
\( m=0, n=4 \)
\( \omega_{a/c^*} = 14.5104 \)

Figure 3. Wavefunctions of the two lowest modes with \( m = 0 \) in a 2D triangular system: the ground state (a), with \( n = 2 \), and the third excited state (b), with \( n = 4 \), at double the frequency of the ground state. Solid/partial shadings represent positive/negative wavefunction values, whose magnitudes correspond to the radii of the symbols. A grid with \( N = 40 \) was only used to present the diagrams. Grid sites without symbols have \(|\psi| < 0.02|\psi_{\text{max}}|\).

\[ \tan \beta = k_2/k_1, \] and the combination goes to zero on boundary \( b_0 \), where \( y = -\frac{a}{2\sqrt{3}} \); thus it is like a standing wave. By assumption, we need \( k_2 \neq 0 \) to have a nonzero wavefunction.

When reflected off the triangle faces \( b_1 \) and \( b_2 \), this simply produces rotations of \( \pm 120^\circ \).

If \( R \) is an operator that rotates vectors through \( +120^\circ \), the additional waves being generated are

\[ \psi_1 = e^{i(R\tilde{k}_1) \cdot \vec{r}} \sin \left[ (R\tilde{k}_2) \cdot \vec{r} + \frac{k_2 a}{2\sqrt{3}} \right], \] \( (4) \)

\[ \psi_2 = e^{i(R^2\tilde{k}_1) \cdot \vec{r}} \sin \left[ (R^2\tilde{k}_2) \cdot \vec{r} + \frac{k_2 a}{2\sqrt{3}} \right]. \] \( (5) \)

In order to determine the allowed \((k_1, k_2)\), one needs to impose DBC on all three boundaries, for a linear combination with unknown coefficients \( A_0, A_1, A_2 \),

\[ \psi = A_0 \psi_0 + A_1 \psi_1 + A_2 \psi_2. \] \( (6) \)

Imposing DBC on all boundaries determines the allowed wavevector components as

\[ k_1 = \frac{2\pi}{3a} m, \quad m = 0, 1, 2... \] \( (7) \)

\[ k_2 = \frac{2\pi}{3a} \sqrt{3} n, \quad n = 1, 2, 3... \] \( (8) \)

Furthermore, the parity constraint \( e^{i\pi m} = e^{i\pi n} \) appears; that is, \( n \) and \( m \) are either both odd or both even.

The resulting \( xy \) frequencies are given by

\[ \omega = c \sqrt{k_1^2 + k_2^2} = \frac{c}{\sqrt{\varepsilon \mu}} \frac{2\pi}{3a} \sqrt{m^2 + 3n^2} \] \( (9) \)
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Figure 4. Wavefunctions of the doubly degenerate first excited state in a 2D triangular system. Two orthogonal sub-states corresponding to different choices of the phase of $\psi$ are presented in (a) and (b). Solid/partial shadings represent positive/negative wavefunction values, whose magnitudes correspond to the radii of the symbols. A grid with $N = 40$ was only used to present the diagrams. Grid sites without symbols have $|\psi| < 0.02|\psi_{\text{max}}|$.

The mode wavefunctions are described completely using the amplitude relationships that result:

$$A_1 = A_0 e^{i\frac{2\pi}{3} m}, \quad A_2 = A_0 e^{-i\frac{2\pi}{3} m}. \quad (10)$$

Therefore, solutions are specified by a choice of integers $m$ and $n$ and the phase of the complex constant $A_0$. In Fig. 2 the frequencies of the lowest modes are presented, scaled with the speed of light in the medium, $c^*$, and arranged as families for each value of the quantum number $n$. This solution is represented in the physically motivated form described by Chang et al. [2] and is entirely equivalent to the first solution given by Lame’ [11] and revisited by various authors [14].

The solutions obtained have obvious symmetry properties. We can get all the possible eigenfrequencies by applying the restrictions, $0 \leq m < n$. Choices with $m \geq n$ or $m < 0$ also give allowed frequencies, however, these only correspond to other $\vec{k}$ rotated by $\pm 120^\circ$ from some original $\vec{k}$ defined with $0 \leq m < n$. For instance, having found $\vec{k} = (k_1, k_2) = \frac{2\pi}{3a}(m, \sqrt{3}n)$, values of $\vec{k}' = (k'_1, k'_2) = \frac{2\pi}{3a}(m', \sqrt{3}n')$ corresponding to $\pm 120^\circ$ rotations result from

$$k'_1 = k_1 \cos 120^\circ \mp k_2 \sin 120^\circ$$
$$k'_2 = \pm k_1 \sin 120^\circ + k_2 \cos 120^\circ \quad (11)$$

which implies transformation to the new mode indexes,

$$m' = -\frac{1}{2} m + \frac{3}{2} n, \quad n' = \pm \frac{1}{2} m - \frac{1}{2} n. \quad (12)$$

As specific examples, the choice $(m, n) = (1, 3)$ gives one mode, which after rotations by $\pm 120^\circ$ could also be described by the integers $(m', n') = (4, 2)$ or by $(m', n') = (5, 1)$. 
This represents only one mode, with different choices of the reference edge or triangle base. Any mode with \( m \neq 0 \) is doubly degenerate; the change \( m \to -m \) gives an independent mode with the same frequency, which rotates in the opposite sense around the triangle. A real representation of the degenerate pairs comes from taking the real and imaginary parts of \( \psi \) to form two independent wavefunctions. Different values of the complex constant \( A_0 \) produce other choices of the two independent wavefunctions. For \( m = 0 \), the wavefunction \( \psi \) can be made pure real; these modes are non-degenerate and \( n \) must be even, since \( m \) and \( n \) must be of equal parity.

The ground state (Fig. 3a) has \( (m, n) = (0, 2) \) and resulting frequency, \( \omega = \frac{c \Delta n}{\sqrt{\varepsilon \mu \sqrt{3} a}} \). This corresponds to a wavelength of \( \lambda = \frac{2\pi}{k} = \frac{\sqrt{3} a}{2} \), which is equal to the height of the triangle, twice as large as naively expected. The wavefunctions of all the modes with \( m = 0 \) have a typical triangular shape (see Fig. 3), periodically patterned over the entire system (invariant under \( \pm 120^\circ \) rotations) and cannot be confined by TIR, see Sec. 3 below. Wavefunctions of the first excited state, which is also the first TIR-confinable mode, are presented in Fig. 4. This and similar states, having nonzero \( m \), are doubly degenerate, therefore, there is some arbitrariness in the wavefunction pictures. Two degenerate wavefunctions were obtained from the real and imaginary parts of \( \psi \) in Equation (6); an equivalent degenerate pair can be obtained by changing the complex constant \( A_0 \). Wavefunctions for the sequence of many of the lowest modes are displayed at www.phys.ksu.edu/~wysin/.

3. TIR confinement and the resonant mode spectrum

The TIR mode confinement can be determined approximately by identifying those modes which completely satisfy the elementary requirements for total internal reflection. We assume that outside the system boundary a different refractive material with relative permeability \( \mu' \) and permittivity \( \varepsilon' \) (which could be vacuum for greatest simplicity) and index \( n' = \sqrt{\varepsilon' \mu'} \), rather than a conducting boundary. If all the plane wave components of a mode satisfy the TIR requirements, then the mode is confined; it should correspond to a resonance of the optical cavity. If any of the plane wave components do not satisfy TIR, the fields of the mode will quickly leak out of the cavity and there should be no resonance at that mode’s frequency (also, without TIR, the assumption of DBC is completely invalidated). These results depend on the index ratio from inside to outside the cavity,

\[
N = \frac{n}{n'} = \frac{\sqrt{\varepsilon \mu}}{\sqrt{\varepsilon' \mu'}}
\]  

(13)

A plane wave of (3D) wavevector \( \vec{K}_i \) has an incident angle on one of the boundaries expressed as

\[
\sin \theta_i = \frac{K_i||}{K_i}
\]  

(14)
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Figure 5. TIR mode confinement limits (index ratio $N = \sqrt{\epsilon / \epsilon'}$) for 2D triangular cavities. Modes are confined where the $m/n$ ratios (as indicated) lie above the solid curve, corresponding to TIR on all boundaries [relation (18)]. Intersections on the $N$-axis give the critical index ratios for each mode.

where $\|$ indicates the component parallel to the boundary. TIR will take place for this wave provided

$$\sin \theta_i > \sin \theta_c = \frac{1}{N}. \quad (15)$$

For 2D geometry, Equation (15) is applied directly to 2D wavevectors, $\vec{K}_i = \vec{k}_i$; one just needs to identify the parallel component of $\vec{k}_i$ relative to a boundary, for each for each plane wave present in $\psi$, Equation (6). The main difficulty is to satisfy TIR on all boundaries on which that component is incident.

For 3D prismatic geometry, the wavevector $\vec{K}_i$ includes a $z$-component. The net parallel component needed in Equation (15) can include planar ($xy$) and vertical ($z$) contributions. The addition of a $z$-component will have the tendency to raise the energy of any mode, and the net wavevector magnitude, thereby improving the possibility for mode confinement.

3.1. Confinement in 2D triangular systems

For both the TM and TE 2D modes of an equilateral triangle, we can use the exact solutions in Sec. 2.1 to investigate whether these can be confined by TIR. Due to the threefold rotational symmetry through angles of $0^\circ$, $120^\circ$ and $-120^\circ$, the TIR condition need only be applied on one of the boundaries, say, the lower boundary $b_0$, parallel to the $x$-axis. The first wave $\psi_0$ is composed of two traveling waves, one incident on $b_0$ and one reflected from $b_0$; both have wavevectors whose $x$-component is $k_1 = \frac{2\pi}{3a} m$, where
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Figure 6. Frequency of the lowest confined mode for a 2D triangular system surrounded by vacuum, as a function of the refractive index. Pairs \((m, n)\) indicate some of the modes’ quantum numbers. No modes are confined for \(\sqrt{\varepsilon \mu} \leq 2\).

\[ m = 0, 1, 2, \ldots \]. The other rotated waves \(\psi_1\) and \(\psi_2\) are composed from traveling waves whose wavevectors have \(x\)-components of magnitudes \(\frac{1}{2}(\sqrt{3}k_2 \pm k_1)\), where \(k_2 = \frac{2\pi}{3a}\sqrt{3n}\) with \(n = 1, 2, 3, \ldots\). For the allowed modes, \(m < n\), or \(k_1 < k_2\), which shows that

\[ k_1 \leq \frac{1}{2}(\sqrt{3}k_2 - k_1) \leq \frac{1}{2}(\sqrt{3}k_2 + k_1). \tag{16} \]

The \(x\)-component of the \(\psi_0\) wave is always the smallest; this wave has the smallest angle of incidence on \(b_0\), so if it undergoes TIR then so do \(\psi_1\) and \(\psi_2\), and the mode is confined. As \(k_1\) is the component parallel to boundary \(b_0\) (i.e., \(k_1||\)), the condition for TIR confinement of the mode is

\[ \sin \theta_i = \frac{k_1}{\sqrt{k_1^2 + k_2^2}} > \sin \theta_c = \frac{1}{N}, \tag{17} \]

or equivalently in terms of quantum numbers \(m\) and \(n\),

\[ \frac{m}{n} > \frac{3}{N^2 - 1}. \tag{18} \]

This relation contains some interesting features. First, since all modes have \(m < n\), the LHS is always less than 1, and confinement of modes can only occur for adequately large refractive index ratio, \(N > 2\). For \(N < 2\) all modes will leak out of the cavity; they cannot be stably maintained by TIR. Secondly, for any particular value \(N > 2\), relation (18) determines a critical \(m/n\) ratio; modes whose \(m/n\) ratio is below the critical value will not be stable. As \(m/n\) relates to the geometrical structure of the mode wavefunction, there is a strong relation between the confined mode structures and the refractive index.
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An alternative way to look at this, is that each particular mode will be confined only if $N$ is greater than a specific critical value determined by the $m/n$ ratio for that mode:

$$N > N_c = \sqrt{\frac{3n^2}{m^2}} + 1. \quad (19)$$

The relation shows that generally speaking, modes with smaller $n/m$ are most easily confined; stated otherwise, the modes where $m$ is closest to $n$ are the ones most readily confined. On the other hand, all modes with $m = 0$ leak out, because these have wave components with a vanishing angle of incidence on the boundaries.

Fig. 5 is a kind of phase diagram for mode confinement by TIR. Some of the modes’ $m/n$ ratios are plotted vs. refractive index ratio $N$, together with relation (18). A particular mode is confined only if its $m/n$ value falls above the critical curve. In this way one can easily see the critical refractive indexes for each mode. Modes where $m$ is close to $n$ require the smallest refractive index values for confinement.

One more application of these results is shown in Fig. 6, where the frequency of the lowest confined mode is plotted versus $\sqrt{\epsilon \mu}$, with vacuum outside the cavity ($\epsilon \mu' = 1$). The quantum numbers $(m, n)$ for the lowest confined mode are indicated in each curve segment. The two different curves correspond to plotting $\omega/c$ (dashed) and frequency scaled with refractive index, $(\omega/c)\sqrt{\epsilon \mu}$ (solid). Of course, as $\sqrt{\epsilon \mu}$ approaches the value 2, the lowest frequency becomes large and the minimizing mode has $m$ very close to $n$, with both large. At the opposite limit of large values of $\sqrt{\epsilon \mu}$, the lowest frequency becomes small, and the mode $(1, 3)$ is always the lowest frequency mode that is confined for any $\sqrt{\epsilon \mu} > 5.29$. The graph only shows the minimum confined frequency at the intermediate values of refractive index $\sqrt{\epsilon \mu}$.

### 3.2. Confinement in triangular based prisms

We consider a vertical prism of height $h$ with a triangular base of edge $a$. With perfectly reflecting end mirrors at $z = 0$ and $z = h$ the allowed longitudinal wavevectors would be $k_z = l\pi/h$, where $l$ is an integer. For a plane wave component with 2D wavevector $\vec{k}_i$, the net 3D wavevector magnitude within the cavity is

$$K_i = \sqrt{k_i^2 + k_z^2}. \quad (20)$$

This increases the incident angles on the walls of the cavity: the presence of nonzero $k_z$ should enhance the possibility for confinement, compared to the purely 2D system.

At the lower and upper ends of the prism, the parallel wavevector component of any of the plane waves present is the full 2D $\vec{k}_i$, which can be expressed as

$$K_{i||}^{(ends)} = |\vec{k}_i| = \sqrt{k_i^2 + k_z^2} = \frac{2\pi}{3a} \sqrt{m^2 + 3n^2}. \quad (21)$$

Finding the incident angle by (14) and Snell’s Law (15), this implies a critical index ratio needed for TIR confinement by the cavity ends,

$$N_{c}^{(ends)} = \frac{1}{\sin \theta_i} = \sqrt{1 + \left(\frac{3}{2\pi}\right)^2 \frac{(k_za)^2}{m^2 + 3n^2}}. \quad (22)$$
On the vertical walls of the prism, the parallel wavevector component is a combination of $k_z$ and the 2D $k_{i\parallel}$. If TIR occurs on one wall then by symmetry it will take place on all the walls. Considering the $b_0$ wall, the $\psi_0$ wave has the largest incident angle as in the 2D problem, and both $k_1$ and $k_z$ determine its parallel wavevector component,

$$K_{i\parallel}^{(\text{walls})} = \sqrt{k_1^2 + k_z^2} = \sqrt{\left(\frac{2\pi}{3a}\right)^2 m^2 + k_z^2}.$$  \hfill (23)

Then this determines the critical index ratio needed for TIR by the cavity walls,

$$N_{c}^{(\text{walls})} = \frac{1}{\sin \theta_i} = \sqrt{1 + \frac{3n^2}{m^2 + \left(\frac{2\pi}{3a}\right)^2(k_z a)^2}}.$$  \hfill (24)

For TIR mode confinement, the actual index ratio $N$ must be greater than both $N_{c}^{(\text{ends})}$ and $N_{c}^{(\text{walls})}$.

Some results are shown in Fig. 7 for the $m = 1$, $n = 3$ modes as a function of $k_z$. The symbols indicate the allowed $k_z a$ values for the prism height equal to the base edge, $h = a$. The limiting 2D critical ratio is seen at the point $k_z a = 0$. For the $m = 1$, $n = 3$ modes, the critical index ratio is lowered from the 2D value $N_{c} \approx 5.29$, down to values as low as $N_{c} \approx 1.40$ for $k_z a \approx 11$. More significantly, the ground state, $m = 0$, $n = 2$, cannot be confined in 2D, however, one finds $N_{c} \approx 1.41$ for $k_z a \approx 7.25$, as seen in Fig. 8. Similar critical index ratio reductions occur for all the other modes.
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Figure 8. Triangular prism critical index ratios as explained in Fig. 7 for the modes with 2D quantum indexes \( m = 0, n = 2 \) (2D ground state).

One sees that \( N_{c}^{\text{(walls)}} \) and \( N_{c}^{\text{(ends)}} \) always cross at some intermediate \( k_{z} \), where they produce the smallest index ratio \( N_{c}^{\text{(min)}} \) needed for TIR. Setting them equal gives

\[
N_{c}^{\text{(min)}} = \sqrt{1 + \left(1 + \frac{m^2}{3n^2}\right)^{-1}},
\]

which occurs at

\[
k_{z}^{*} = \left(\frac{2\pi}{3a}\right) \sqrt{3n} = k_{2}.
\]

It is a rather intriguing result; choosing \( h/a \) such that a mode will occur at this \( k_{z} \) will be the optimum choice for having the mode confined most easily by TIR. This could be useful for control over selection of desired modes in a cavity.

Cautionary comments are in order. The above discussion applies exactly only to a scalar wave. For electromagnetic waves, it applies to either the TM or TE polarizations in an approximate sense for quasi-2D EM modes, requiring small longitudinal wavevector \( k_{z}a < 1 \). The greatest enhancements in TIR were found to occur at values \( k_{z}a > 1 \), primarily because a reasonably large \( k_{z} \) is needed in order to satisfy the TIR requirements at the ends. The presence of significant \( k_{z} \) values, however, will lead to a mixing of the TM and TE polarizations, contrary to the original assumptions. Then, for practical purposes, the calculation is mostly interesting for how it indicates the improvement in TIR confinement expected mainly on the 2D walls of the cavity, at small \( k_{z} \).
4. Conclusions

The resonant modes of an optical cavity with an equilateral 2D cross-section have been examined in an approximate manner, starting from the exact analytic solutions for Dirichlet boundary conditions. The six plane waves that make up each mode have been analyzed for the conditions necessary such that all are confined in the cavity by TIR. For 2D electromagnetics \((k_z = 0)\), modes with larger ratios of quantum indexes \(m/n\) are most easily confined; conversely, modes with \(m = 0\) have wavefunctions invariant under \(\pm 120^\circ\) rotations and never undergo TIR confinement. The critical index ratios for TIR confinement are the same for TM and TE polarizations. For a 3D prism, a nonzero longitudinal wavevector \(k_z\) improves the possibilities for TIR confinement and offers a way to control mode selection.

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