Classical Probability

Goal
To gain an understanding of using probabilistic representations such as the probability density to study physical systems.

Introduction

In quantum mechanics the behavior of electrons and other small particles is not Newtonian. In fact, the behavior of quantum systems can almost appear random. Instead of having predictable trajectories that we expect of projectiles in Newtonian mechanics, individual quantum events are unpredictable. However, if we consider a large number of quantum events, a pattern emerges. This is known as a statistical distribution. We often use mathematical probabilities to describe the outcomes of quantum events. The probability of a single quantum event is exactly proportional to the value of the statistical distribution.

In this tutorial we will reacquaint ourselves with the ideas of classical probability so we can confidently use those ideas in quantum systems.

A. Balls On Tracks

Consider the experiment shown in Figure 1. A series of balls is set rolling toward the right at a *small velocity* v_0. In your calculations, ignore this initial velocity as well as the effects of friction.

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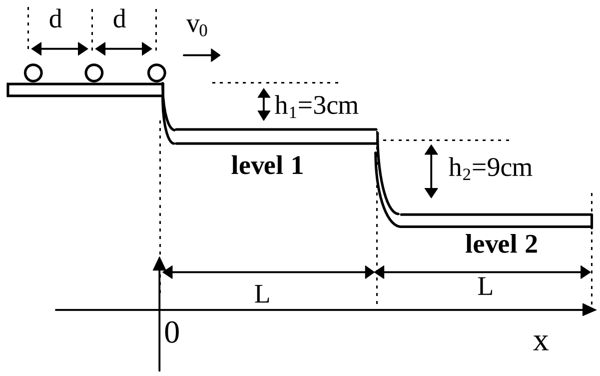


Figure 1. Diagram of the balls on tracks experiment

A-1. Describe the speed of the ball throughout its motion, and sketch velocity vs. position (v vs. x) from 0 to 2L on the graph provided.

A-2. Determine the ratio v_2/v_1, where v_1 is the speed of the ball on level 1 and v_2 is the speed on level 2. Explain how you arrived at your answer.

A-3. Determine the ratio t_2/t_1, where t_1 is the time the ball spends on level 1 and t_2 is the time it spends on level 2. Explain how you arrived at your answer.

Suppose balls are repeatedly set in motion so that at the instant a ball leaves level 2, another ball is released onto level 1. This creates a periodic motion. Define T as the total time for a ball to run through the two levels.

A-4. Find T in terms of L and v_1.

A-5. If you are taking pictures of the ball at random times, will there be more pictures showing the ball on level 1 or on level 2? Why?

A-6. Find $\boldsymbol{P}_1$, the probability of finding the ball on level 1, and $\boldsymbol{P}_2$, the probability of finding the ball on level 2.

For the next several questions assume that the lengths of level 1 and level 2 are L_1 and L_2 where $L_1 \neq L_2$.

A-7. Find the period, T, in terms of L_1, L_2 and v_1. Explain how you arrived at your answer.

A-8. Find P_1, the probability of finding a ball on level 1, and P_2, the probability of finding it on level 2 in terms of T, v_1 or v_2, and L_1 or L_2.

A-9. How much time does the ball spend between x_1 and $x_1 + \Delta x$, where x_1 is between 0 and L_1 and Δx is small?

A-10. Use your answer from A-9 to find $P(x_1, \Delta x)$, the probability of finding the ball between x_1 and $x_1 + \Delta x$.

A-11. Find $P(x_1, \frac{\Delta x}{2})$, the probability of finding the ball between x_1 and x_1 + .

A-12. Make comparisons for:

- $\boldsymbol{P}(x_1, \Delta x) / \Delta x$ and $\boldsymbol{P}(x_1, \frac{\Delta x}{2} / \frac{\Delta x}{2}$.

- $\boldsymbol{P}(x_1, \Delta x)$, $\boldsymbol{P}(x_1, \frac{\Delta x}{2})$, $\boldsymbol{P}(x_2, \Delta x)$, and $\boldsymbol{P}(x_2, \frac{\Delta x}{2})$, where x_1 is between 0 and L_1 and x_2 is between L_1 and $L_1 + L_2$.

- $\boldsymbol{P}(x_1, \Delta x) / \Delta x$ and $\boldsymbol{P}(x_2, \Delta x) / \Delta x$

Congratulations! You have just discovered the equation for *Probability density* which is defined as $P(x) = \boldsymbol{P}(x, dx) / dx$.

Now consider the case where the lengths of level 1 and level 2 are both equal to L.

A-13. In the space below, sketch a graph of P(x) vs. x. How, if at all, does your answer depend on the value of dx? Explain your reasoning.

A-14 What is the probability of finding a ball between 0 and 2L? Find an expression to represent this condition in terms of P(x).

B. Glider and Springs

Consider the experiment shown in Figure 2. A glider sits on an air track and is attached at either end with two identical springs. Assume the glider moves without friction and that the track is level.

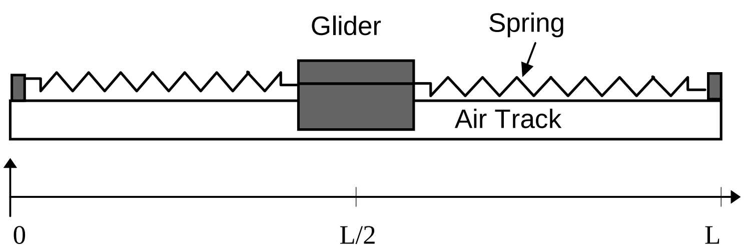
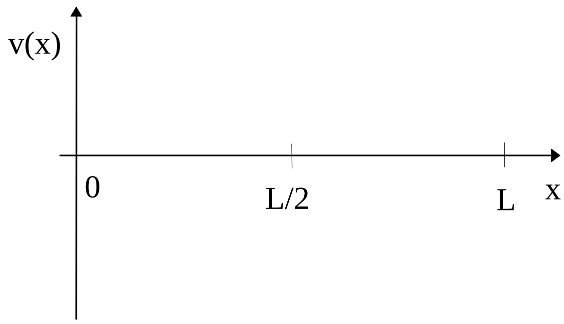


Figure 2. Diagram of the glider and springs experiment

B-1. Describe the motion of the glider and discuss at what locations on the track the glider is moving faster and slower.

B-2. On the graph provided, sketch the speed of the glider as a function of x. Then divide the range of motion of the glider into 8 regions of equal length and separate them with vertical lines on your graph.



B-3. What determines the probability that a randomly taken picture will show the glider in a given region? Explain your reasoning.

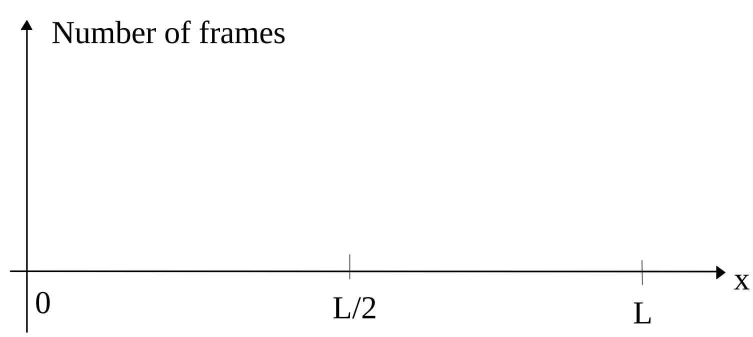
Suppose the whole setup is in a dark room with a small, randomly flashing light bulb attached to the center of the glider. A photograph of the glider is then taken with an exposure time much longer than the period of oscillation allowing hundreds of flashes to be imprinted on the film as bright dots.

B-4. Predict what the picture will look like. Discuss your results with your group and then ask your instructor to see a picture and verify your results.

Video analysis software enables you to study moving objects by allowing you to locate, on a digital video, the position of the object in any given time frame and thus collect data.

B-5. Suppose you are given a list, from the analysis of a video, of the position of the glider in each frame of a complete cycle and you are to randomly select frames and record the position of the glider in each frame. You can make multiple selections of the same frame but you must give each frame an equal chance of being selected. You are to continue selecting frames until you have a large collection of data points. From your hypothetical collection, predict where you might have more (and less) records of glider positions. Explain why.

B-6. Based on your prediction, sketch the expected relationship between the numbers of frames you select and the position of the glider in each frame.



B-7. On the computer, start your video analysis program. Ask your instructor if you need assistance. Open the movie "Oscillator" and use the software to find the position of the glider in each frame. Import the data into the spreadsheet "Classical Probability Worksheet". This spreadsheet can help you generate random numbers and pick appropriate frames based on the random numbers. Try a few see how it works and let the computer to do a large number (>1000). Then plot the results. Is the graph consistent with your predictions?

C. The General Formula

Consider an object oscillating with a period T and amplitude A.

C-1. If the object spends dt in an interval dx, what is the *probability* that a randomly taken picture will show the object in the interval dx?

C-2. Show that the *probability density* of the object as a function of x is given by

(1)